Using Classical Probability To Guarantee Properties of Infinite Quantum Sequences

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Abstract

We consider the product of infinitely many copies of a spin-$\frac{1}{2}$ system. We construct projection operators on the corresponding nonseparable Hilbert space which measure whether the outcome of an infinite sequence of σ^x measurements has any specified property. In many cases, product states are eigenstates of the projections, and therefore the result of measuring the property is determined. Thus we obtain a nonprobabilistic quantum analogue to the law of large numbers, the randomness property, and all other familiar almost-sure theorems of classical probability.

By studying infinitely many copies of a quantum system, it is possible to eliminate references to probability from the postulates of quantum mechanics. It suffices to consider an infinite product of spin-$\frac{1}{2}$ systems. Statements of the form "In a $\widehat{\sigma}^z$-eigenstate, the probability that a single σ^x-measurement will yield $+1$ is $\frac{1}{2}$" can be replaced by "If the state is an infinite product of $\widehat{\sigma}^z$-eigenstates, half of the σ^x-measurements will yield $+1$" [1].

Recently Coleman and Lesniewski [2] constructed a randomness operator, using the classical notion of randomness of a sequence of $+1$'s and -1's made

precise by Kolmogorov and Martin-Löf. Their operator measures whether a sequence of independent σ^x-measurements yields a random sequence of $+1$'s and -1's. If the state is a product of $\widehat{\sigma}^z$-eigenstates, they show that the answer is **yes**.

In this note, we construct a quantum operator which measures whether the outcome of the infinite sequence of σ^x-measurements has **any** given property. We use the spectral theorem to provide an automatic construction of the operator, which is a projection.

Corresponding to each quantum state in the product Hilbert space, there is a classical probability measure on the set of sequences of +1's and -1's. We show how to use this measure to determine how the operator acts on the state. For many cases of interest, the classical probability of the set of sequences which have the property is 0 or 1. We will show that this implies that the state is an eigenstate of the operator, with eigenvalue 0 or 1. That is, if the classical probability is 1, the state **has** the property. If the classical probability is 0, the state doesn't have it. For example if the state is a product of $\widehat{\sigma}^z$-eigenstates, the corresponding measure is the one associated with independent fair coin flips. If for sequences generated by this coin flip process the property holds with probability 1, then the operator will assert that the state has the property.

Furthermore, suppose the property is a "tail event", meaning that for each n, it can be described without referring to the first n spins. Then any product state must be an eigenstate of the corresponding projection, and again the question of whether the state has the property has a definite, probability-free answer.

Let V be a two dimensional Hilbert space, and denote the $\widehat{\sigma}^x$ basis by $|+1\rangle$ and $|-1\rangle$. To describe a sequence of independent copies of V, we form the tensor product $V^\infty = V \otimes V \otimes \cdots$, which is a nonseparable Hilbert space [3]. The operator $\widehat{\sigma}^x_n$ corresponds to measuring σ^x in the nth copy; for any $|\psi_j\rangle \epsilon V$, with $\langle\psi_j|\psi_j\rangle = 1$, the product state $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$ is in V^∞, and

$$\begin{aligned} \widehat{\sigma}^x_n\,(|\psi_1\rangle \otimes \cdots |\psi_{n-1}\rangle \otimes |\pm 1\rangle \otimes |\psi_{n+1}\rangle \otimes \cdots) \\ = \pm\,(|\psi_1\rangle \otimes \cdots |\psi_{n-1}\rangle \otimes |\pm 1\rangle \otimes |\psi_{n+1}\rangle \otimes \cdots) \end{aligned} \tag{1}$$

The classical probability analogue to this quantum system is the (uncount-

able) set of sequences of +1's and -1's, which we will denote by Ω. "Properties" of the sequences are subsets of Ω. For example, the set of sequences with the (probability $\frac{1}{2}$) law-of-large-numbers property is

$$\left\{(\sigma_1, \sigma_2, \cdots) \;:\; \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sigma_n = 0\right\} \tag{2}$$

Now we use the spectral theorem to construct projections from subsets. Given a subset of Ω, let $F(\sigma_1, \sigma_2, \cdots)$ denote its indicator function, i. e. $F(\sigma_1, \sigma_2, \cdots) = 1$ if $(\sigma_1, \sigma_2 \cdots)$ is in the subset and $F = 0$ if not. The spectral theorem then implies that the projection operator

$$\widehat{F} = F(\widehat{\sigma}_1^x, \widehat{\sigma}_2^x, \cdots) \tag{3}$$

is uniquely defined on all of V^∞. The nonseparability of V^∞ is no impediment. The subsets must be Borel, a condition that doesn't depend on any particular measure that may be defined on Ω. Any subset you can describe (without using the Axiom of Choice) is Borel.

By using the spectral theorem (and thus taking advantage of the work done in proving it) we can directly define $\widehat{F}$ by (3), and can avoid considering limits of sequences of approximations to the desired operators, which are used in previous approaches [2, 4, 5].

On states which are simultaneous $\widehat{\sigma}_n^x$-eigenstates for all n, we have

$$\widehat{F}\left(|\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \cdots\right) = F\left(\sigma_1, \sigma_2, \cdots\right)\left(|\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \cdots\right) \tag{4}$$

There are, however, many states in V^∞ which are not superpositions of these $\widehat{\sigma}_n^x$-eigenstates, and we want to know how $\widehat{F}$ acts on them. In particular, for states of the form $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$, which do constitute an overcomplete basis for V^∞, we want to know $\|\widehat{F}\left(|\psi_1\rangle \otimes |\psi_2\rangle \cdots\right)\|^2$.

First consider the more familiar case of an operator with a continuous spectrum, for example, the position operator $\widehat{x}$. With $\widehat{G} = G(\widehat{x})$ we can write

$$\|\widehat{G}|\psi\rangle\|^2 = \int |G(x)|^2 \;\; |\langle x|\psi\rangle|^2 dx \tag{5}$$

Note that the measure given by $d\mu = |\langle x|\psi\rangle|^2 dx$ depends on $|\psi\rangle$ but not on G.

Similarly, with $\widehat{G} = G\left(\widehat{\sigma}_1^x, \widehat{\sigma}_2^x, \cdots\right)$ we have

$$\|\widehat{G}(|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots)\|^2 = \int |G(\sigma_1, \sigma_2, \cdots)|^2 d\mu(\sigma_1, \sigma_2, \cdots) \tag{6}$$

for any bounded Borel-measurable G. (The cases of (5) and (6) are not completely analogous. In (5), as G varies, $\widehat{G}|\psi\rangle$ can span all of the Hilbert space. In (6), the span of $\widehat{G}(|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots)$ can only be a (separable) subspace of V^∞ called the component of $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$).

As in (5), the measure μ in (6) depends on $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$ but not on G. We'll now determine μ in terms of $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$ by choosing G to be indicator functions depending only on finitely many $\widehat{\sigma}_n^x$. For indicator functions F, formula (6) becomes

$$\|\widehat{F}\left(|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots\right)\|^2 = \int F\left(\sigma_1, \sigma_2, \cdots\right) d\mu(\sigma_1, \sigma_2, \cdots) \tag{7}$$

First consider the case of one factor of V. Let $\langle\psi_1|\psi_1\rangle = 1$. As an example, let $f(\sigma_1) = 1$ if $\sigma_1 = +1$ and 0 if $\sigma_1 = -1$. Then $\|\widehat{f}|\psi_1\rangle\|^2 = \|f\left(\widehat{\sigma}_1^x\right)|\psi_1\rangle\|^2 = \|\langle +1|\psi_1\rangle\|^2$. For any of the four possible indicator functions f we can write

$$\|\widehat{f}|\psi_1\rangle\|^2 = \int f(\sigma_1) d\mu(\sigma_1) \tag{8}$$

where the measure μ assigns point masses to +1 and –1 by $\mu(+1) = |\langle +1|\psi_1\rangle|^2 = 1 - |\langle -1|\psi_1\rangle|^2 = 1 - \mu(-1)$. This follows from how $f\left(\widehat{\sigma}_1^x\right)$ acts on eigenstates.

Similarly in the V^∞ case, given a state $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$ with each $\langle\psi_j|\psi_j\rangle = 1$, let $|\langle +1|\psi_j\rangle|^2 = p_j$. (This definition of p_j makes explicit that the measure depends on the state $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$ and on the decision to measure σ^x.) Let μ be the measure on Ω which assigns probability p_j to the set of sequences with $\sigma_j = +1$, and for which the coordinates $\sigma_1, \sigma_2, \cdots \sigma_n$ are independent for any n, i. e.

$$\begin{aligned} &\mu\left\{(\sigma_1, \sigma_2, \cdots) : \sigma_1 = \omega_1, \sigma_2 = \omega_2, \cdots, \sigma_n = \omega_n\right\} \\ &\quad = p_1^{\omega_1}(1 - p_1)^{1-\omega_1} \cdots p_n^{\omega_n}(1 - p_n)^{1-\omega_n} \end{aligned} \tag{9}$$

This expression for μ follows from repeating the one-dimensional argument for n dimensional functions. Probability theory ensures that μ is completely determined by (9).

We are now equipped to connect directly classical and quantum statements. For example, consider the (probability $\frac{1}{2}$) law-of-large-numbers property: $F(\sigma_1, \sigma_2, \cdots) = 1$ if $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sigma_n = 0$ and $F = 0$ if not. If $|\psi_n\rangle$ is a $\widehat{\sigma}^z$-eigenstate for each n, say $|\psi_n\rangle = \frac{1}{\sqrt{2}}|+1\rangle + \frac{1}{\sqrt{2}}|-1\rangle$, then $p_n = \frac{1}{2}$ for each n. The classical strong law of large numbers asserts $\int F(\sigma_1, \sigma_2, \cdots) d\mu(\sigma_1, \sigma_2, \cdots) = 1$ and by (7) we have $\|\widehat{F}\left(|\psi_1\rangle \otimes |\psi_2 \otimes \cdots\right)\|^2 = 1$, implying

$$\widehat{F}\left(|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots\right) = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \tag{10}$$

Thus if we measure the law-of-large-numbers projector on this product of $\widehat{\sigma}^z$-eigenstates, we obtain 1.

Randomness works the same way. $F(\sigma_1, \sigma_2, \cdots) = 1$ if $(\sigma_1, \sigma_2, \cdots)$ has the Kolmogorov-Martin-Löf randomness property. Use the same $|\psi_n\rangle$ as above, so p_n and μ remain the same. Again, classical probability says that according to μ, almost every sequence $(\sigma_1, \sigma_2, \cdots)$ is random [6]. So $\widehat{F}(|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots) = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$ where now $\widehat{F}$ is the randomness operator. Of course, the same conclusion holds for any property which has probability 1 according to μ. Analogous results hold for different $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$ which give rise to different μ.

Properties such as the law of large numbers and randomness are tail events; that is, for each n, they depend only on $\sigma_{n+1}, \sigma_{n+2}, \cdots$ (Clearly $\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \sigma_m = \lim_{N\to\infty} \frac{1}{N} \sum_{m=n+1}^{\infty} \sigma_m$. See [6] for details about randomness that show it is a tail event.) The classical Kolmogorov zero-one law [7] asserts that if F is the indicator function of a tail event, and μ is determined by $p_1, p_2, \cdots$ as in (7), then

$$\int F\left(\sigma_1, \sigma_2, \cdots\right) d\mu\left(\sigma_1, \sigma_2, \cdots\right) = 0 \,\text{or}\, 1 \tag{11}$$

Thus for any $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$ and F corresponding to a tail event, $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$ must be an eigenstate of the projection $\widehat{F}$. Every product state $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$ either **has** or **doesn't have** any given tail property.

So product states are eigenstates of projection operators corresponding to many interesting properties. The derivation of this fact and the calculation of the eigenvalue requires classical probability theory, but the quantum statement makes no reference to probability.

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